An introduction to the scaling limits of random walks via the resistance metric

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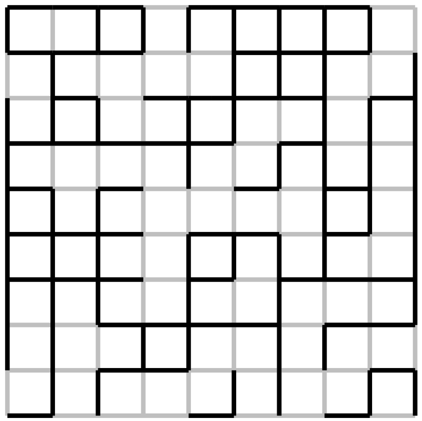
1. MOTIVATION

RANDOM WALK ON A PERCOLATION CLUSTER

Bond percolation on integer lattice $\mathbb{Z}^d$ $(d \geq 2)$, parameter $p \in (0,1)$. E.g. $p = 0.53$,

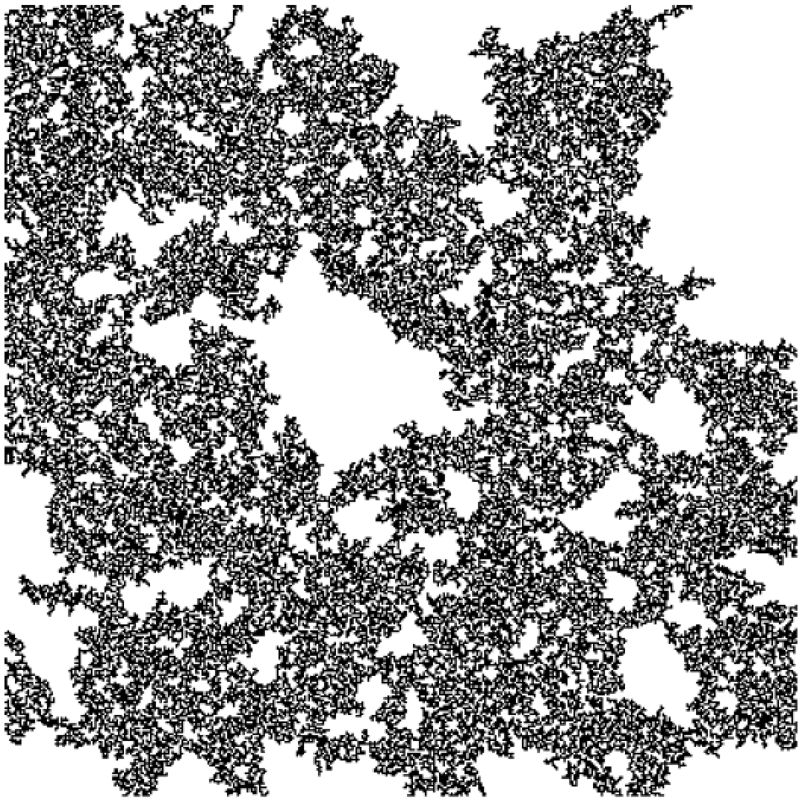


If $p > p_c(d)$, then the random walk is diffusive for $\mathbf{P}$-a.e. environment. In particular,

$$\left(n^{-1} X^{\mathcal{C}}_{tn^2}\right)_{t \geq 0} \to \left(B_{c(d,p)t}\right)_{t \geq 0}.$$

See [Sidoravicius/Sznitman 2004, Biskup/Berger 2007, Mathieu/Piatnitski 2007], and also heat kernel estimates of [Barlow 2004].

PERCOLATION AT CRITICALITY?



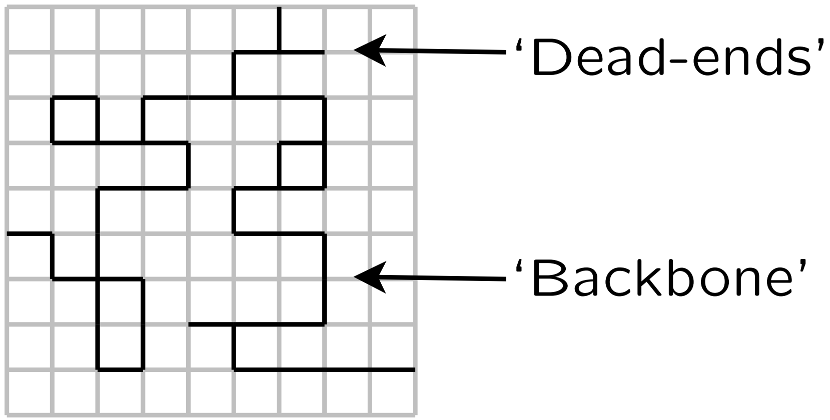
Part of the (near-)critical percolation infinite cluster. Source: Ben Avraham/Havlin.

INCIPIENT INFINITE CLUSTER

At $p = p_c(d)$, it is partially confirmed that there is no infinite cluster. Instead, study the random walk on the 'incipient infinite cluster':

$$\mathcal{C}_0 | \{|\mathcal{C}_0| = n\} \to \mathrm{IIC}.$$

Constructed in [Kesten 1986] for $d = 2$, [van der Hofstad/Jarai 2004] for high dimensions.



Tree-like in high dimensions [Hara/Slade 2000], see also [Heydenreich, van der Hofstad/Hulsfhof/Miermont 2017].

SRW ON PERCOLATION AT CRITICALITY?

Random walk is subdiffusive for $d = 2$ and in high-dimensions [Kesten 1986, Nachmias/Kozma 2009], see also [Heydenreich/ van der Hofstad/Hulshof 2014].

For example, for almost-every-realisation of the IIC in high-dimensions, we have:

$$\frac{\log E_0^{IIC} \tau(R)}{\log R} \to 3,$$

where $\tau(R) = \inf\{n : \, d_{IIC}(0, X_n^{IIC}) = R\}$, and

$$\frac{\log E_0^{IIC} \tilde{\tau}(R)}{\log R} \to 6,$$

where $\tilde{\tau}(R) = \inf\{n : \, |0 - X_n^{IIC}| = R\}$.

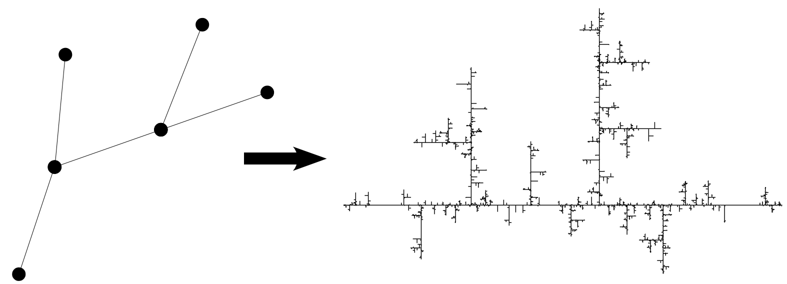
Scaling limit?

E.G. CRITICAL GALTON-WATSON TREES

Let T_n be a Galton-Watson tree with a critical (mean 1), aperiodic, finite variance offspring distribution, conditioned to have n vertices, then

$$n^{-1/2}T_n \to \mathcal{T},$$

where $\mathcal{T}$ is (up to a constant) the **Brownian continuum random tree (CRT)** [Aldous 1993], also [Duquesne/Le Gall 2002].



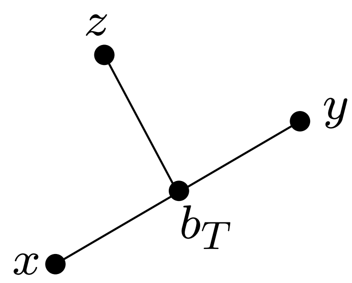
Convergence in Gromov-Hausdorff-Prohorov topology implies

$$\left(n^{-1/2}X^{T_n}_{n^{3/2}t}\right) \to \left(X^{\mathcal{T}}_t\right)_{t\geq 0},$$

see [Krebs 1995], [C. 2008] and [Athreya/Löhr/Winter 2014].

SOME INTUITION

Suppose T is a graph tree, and X^T is the discrete time simple random walk on T, $\pi(\{x\}) = \deg_T(x)$ its invariant measure. The following two properties are then easy to check:



- **[Scale]** For $x, y, z \in T$,

$$P_z^T\left(\sigma_x < \sigma_y\right) = \frac{d_T(b_T(x,y,z),y)}{d_T(x,y)}.$$

- **[Speed]** Expected number of visits to z when started at x and killed at y,

$$d_T(b_T(x,y,z),y)\pi(\{z\}).$$

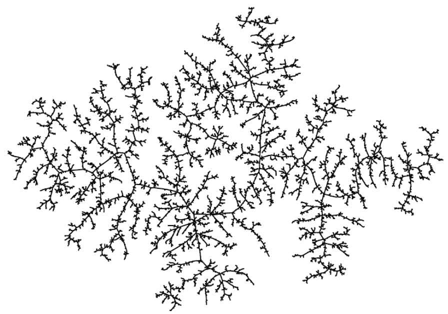
Analogous properties hold for limiting diffusion.

cf. One-dimensional convergence results of [Stone 1963].

OTHER INTERESTING EXAMPLES

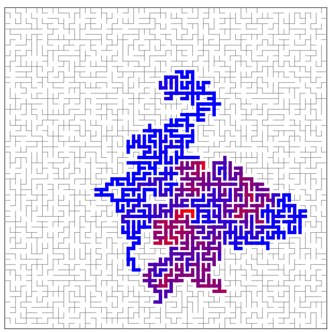
[Critical random graph] For largest connected component $\mathcal{C}_1^n$ of $G(n,1/n)$:

$$\left(\mathcal{C}_1^n, n^{-1/3}R_n, n^{-2/3}\mu_n\right) \to (F, R, \mu),$$



cf. [Addario-Berry, Broutin, Goldschmidt 2012]. We will show it follows that

$$\left(n^{-1/3}X_{tn}^n\right)_{t\geq 0} \to (X_t)_{t\geq 0}.$$



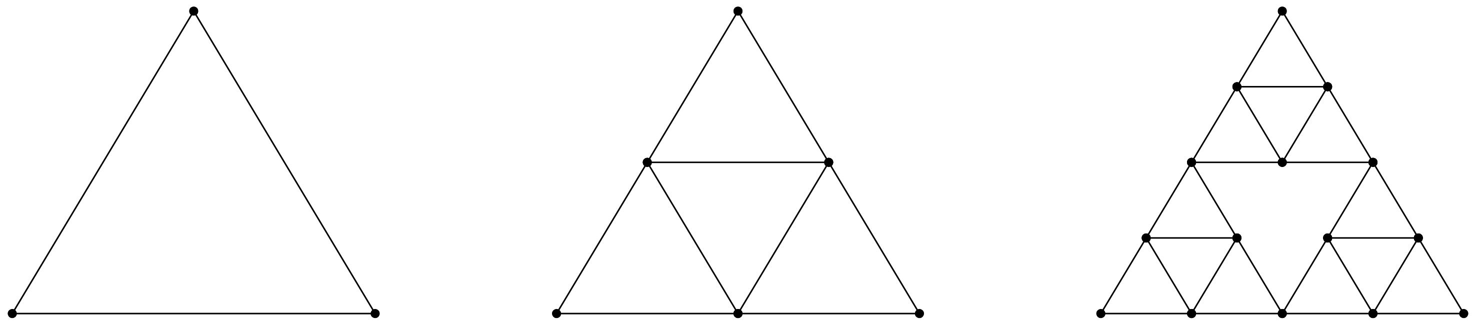
[Uniform spanning tree in two dimensions] Can check that:

$$\left(n^{-5/4}X_{tn^{13/4}}^{UST}\right)_{t\geq 0} \to (X_t)_{t\geq 0}.$$

SELF-SIMILAR FRACTALS

Many of the techniques we will see are useful for random graphs/ fractals were developed for self-similar ones. E.g. [Barlow/ Perkins 1988] constructed a diffusion on the Sierpinski gasket via approximation by SRW:

$$\left(2^{-n} X^n_{t5^n}\right)_{t\geq 0} \to (X_t)_{t\geq 0}.$$



This result can also be understood via the resistance metric, e.g. [Kigami 2001].

RANDOM CONDUCTANCE MODEL AND BOUCHAUD TRAP MODEL

Random conductance model (RCM):

Equip edges of graphs with random weights $(c(x,y))$ such that

$$P(c(x,y) \geq u) = u^{-\alpha}, \qquad \forall u \geq 1,$$

for some $\alpha \in (0,1)$. Subdiffusive scaling limit for associated RW on the integer lattice [Barlow/Cerny 2011, Cerny 2011].

Symmetric Bouchaud trap model (BTM):

Add exponential holding times, mean τ_x, to vertices. In the case where τ is random and heavy-tailed, behaviour similar to RCM.

OUTLINE

[and references]

1. Motivation

2. Random walks and the resistance metric on finite graphs
[Doyle/Snell 1984, Levin/Peres/Wilmer 2009, Lyons/Peres 2016]

3. Stochastic processes associated with resistance metrics
[Kigami 2001, 2012]

4. Convergence results
[C./Hambly/Kumagai 2017, C. 2017+]

5. Applications

RANDOM WALKS ON GRAPHS

Let $G = (V, E)$ be a finite, connected graph, equipped with (strictly positive, symmetric) edge conductances $(c(x,y))_{\{x,y\}\in E}$. Let μ be a finite measure on V (of full-support).

Let X be the continuous time Markov chain with generator Δ, as defined by:

$$(\Delta f)(x) := \frac{1}{\mu(\{x\})} \sum_{y:\, y\sim x} c(x,y)(f(y) - f(x)).$$

NB. Common choices for μ are:

- $\mu(\{x\}) := \sum_{y:\, y\sim x} c(x,y)$, the **constant speed random walk (CSRW)**;
- $\mu(\{x\}) := 1$, **the variable speed random walk (VSRW)**.

DIRICHLET FORM AND RESISTANCE METRIC

Define a quadratic form on G by setting

$$\mathcal{E}(f,g) = \frac{1}{2} \sum_{x,y:x\sim y} c(x,y)\,(f(x)-f(y))\,(g(x)-g(y))\,.$$

Note that (regardless of the particular choice of μ,) $\mathcal{E}$ is a **Dirichlet form** on $L^2(\mu)$, and

$$\mathcal{E}(f,g) = -\sum_{x\in V} (\Delta f)(x) g(x) \mu(\{x\}).$$

Suppose we view G as an electrical network with edges assigned conductances according to $(c(x,y))_{\{x,y\}\in E}$. Then the **effective resistance** between x and y is given by

$$R(x,y)^{-1} = \inf\left\{\mathcal{E}(f,f) :\ f(x)=1, f(y)=0\right\}.$$

R is a metric on V, e.g. [Tetali 1991], and characterises the weights (and therefore the Dirichlet form) uniquely [Kigami 1995].

SUMMARY

RANDOM WALK X WITH GENERATOR Δ

$$\Updownarrow$$

DIRICHLET FORM $\mathcal{E}$ on $L^2(\mu)$

$$\Updownarrow$$

RESISTANCE METRIC R AND MEASURE μ

RESISTANCE METRIC, e.g. [KIGAMI 2001]

Let F be a set. A function $R : F \times F \to \mathbb{R}$ is a **resistance metric** if, for every finite $V \subseteq F$, one can find a weighted (i.e. equipped with conductances) graph with vertex set V for which $R|_{V \times V}$ is the associated effective resistance.

EXAMPLES

- Effective resistance metric on a graph;

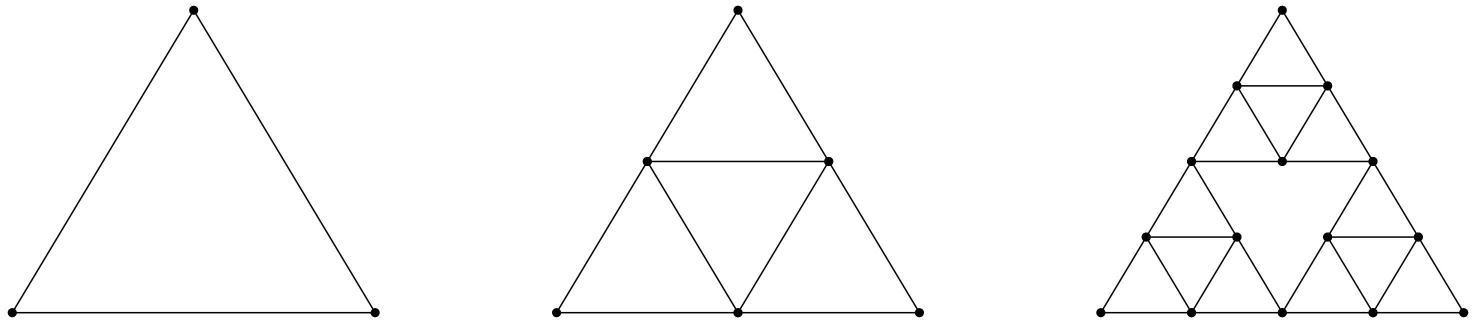
- One-dimensional Euclidean (not true for higher dimensions);

- Any shortest path metric on a tree;

- Resistance metric on a Sierpinski gasket, where for ‘vertices’ of limiting fractal, we set

$$R(x, y) = (3/5)^n R_n(x, y),$$

then use continuity to extend to whole space.



RESISTANCE AND DIRICHLET FORMS

Theorem (e.g. [Kigami 2001]) There is a one-to-one correspondence between resistance metrics and a class of quadratic forms called **resistance forms**.

The relationship between a resistance metric R and resistance form $(\mathcal{E},\mathcal{F})$ is characterised by

$$R(x,y)^{-1}=\inf\left\{\mathcal{E}(f,f):\ f\in\mathcal{F},\ f(x)=1,\ f(y)=0\right\}.$$

Moreover, if (F,R) is compact, then $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form on $L^2(\mu)$ for any finite Borel measure μ of full support. (Version of the statement also hold for locally compact spaces.)

RESISTANCE FORM DEFINITION, e.g. [KIGAMI 2012]

[RF1] $\mathcal{F}$ is a linear subspace of the collection of functions $\{f : F \to \mathbb{R}\}$ containing constants, and $\mathcal{E}$ is a non-negative symmetric quadratic form on $\mathcal{F}$ such that $\mathcal{E}(f,f) = 0$ if and only if f is constant on F.
[RF2] Let $\sim$ be the equivalence relation on $\mathcal{F}$ defined by saying $f \sim g$ if and only if $f - g$ is constant on F. Then $(\mathcal{F}/\sim, \mathcal{E})$ is a Hilbert space.
[RF3] If $x \neq y$, then there exists an $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.
[RF4] For any $x, y \in F$,

$$\sup\left\{\frac{|f(x) - f(y)|^2}{\mathcal{E}(f,f)} :\ f \in \mathcal{F},\ \mathcal{E}(f,f) > 0\right\} < \infty.$$

[RF5] If $\bar{f} := (f \wedge 1) \vee 0$, then $f \in \mathcal{F}$ and $\mathcal{E}(\bar{f},\bar{f}) \leq \mathcal{E}(f,f)$ for any $f \in \mathcal{F}$.

SUMMARY

RESISTANCE METRIC R AND MEASURE μ

$$\updownarrow$$

RESISTANCE FORM $(\mathcal{E}, \mathcal{F})$, DIRICHLET FORM on $L^2(\mu)$

$$\updownarrow$$

STRONG MARKOV PROCESS X WITH GENERATOR Δ,
where

$$\mathcal{E}(f,g) = -\int_F (\Delta f) g d\mu.$$

A FIRST EXAMPLE

Let $F = [0,1]$, $R =$ Euclidean, and μ be a finite Borel measure of full support on $[0,1]$. Define

$$\mathcal{E}(f,g) = \int_0^1 f'(x)g'(x)dx, \qquad \forall f, g \in \mathcal{F},$$

where $\mathcal{F} = \{f \in C([0,1]) : f \text{ is abs. cont. and } f' \in L^2(dx)\}$. Then $(\mathcal{E},\mathcal{F})$ is the resistance form associated with $([0,1], R)$. Moreover, $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form on $L^2(\mu)$. Note that

$$\mathcal{E}(f,g) = -\int_0^1 (\Delta f)(x)g(x)\mu(dx), \qquad \forall f \in \mathcal{D}(\Delta),\ g \in \mathcal{F},$$

where $\Delta f = \frac{d}{d\mu}\frac{df}{dx}$, and $\mathcal{D}(\Delta)$ contains those f such that: f' exists and df' is abs. cont. w.r.t. μ, $\Delta f \in L^2(\mu)$, and $f'(0) = f'(1) = 0$.

If $\mu(dx) = dx$, then the Markov process naturally associated with Δ is reflected Brownian motion on $[0,1]$.

3. CONVERGENCE OF RESISTANCE METRICS AND STOCHASTIC PROCESSES

MAIN RESULT [C. 2016]

Write $\mathbb{F}_c$ for the space of marked compact resistance metric spaces, equipped with finite Borel measures of full support. Suppose that the sequence $(F_n, R_n, \mu_n, \rho_n)_{n\geq 1}$ in $\mathbb{F}_c$ satisfies

$$(F_n, R_n, \mu_n, \rho_n) \to (F, R, \mu, \rho)$$

in the (marked) Gromov-Hausdorff-Prohorov topology for some $(F, R, \mu, \rho) \in \mathbb{F}_c$.

It is then possible to isometrically embed $(F_n, R_n)_{n\geq 1}$ and (F, R) into a common metric space (M, d_M) in such a way that

$$P^n_{\rho_n}\left((X^n_t)_{t\geq 0} \in \cdot\right) \to P_\rho\left((X_t)_{t\geq 0} \in \cdot\right)$$

weakly as probability measures on $D(\mathbb{R}_+, M)$.

Holds for locally compact spaces if $\limsup_{n\to\infty} R_n(\rho_n, B_{R_n}(\rho_n, r)^c)$ diverges as $r \to \infty$. (Can also include 'spatial embeddings'.)

PROOF IDEA 1: RESOLVENTS

For $(F, R, \mu, \rho) \in \mathbb{F}_c$, let

$$G_x f(y) = E_y \int_0^{\sigma_x} f(X_s) ds$$

be the resolvent of X killed on hitting x. NB. Processes associated with resistance forms hit points.

We have [Kigami 2012] that

$$G_x f(y) = \int_F g_x(y,z) f(z) \mu(dz),$$

where

$$g_x(y,z) = \frac{R(x,y) + R(x,z) - R(y,z)}{2}.$$

Metric measure convergence $\Rightarrow$ resolvent convergence $\Rightarrow$ semigroup convergence $\Rightarrow$ finite dimensional distribution convergence.

PROOF IDEA 2: TIGHTNESS

Using that X has local times $(L_t(x))_{x\in F, t\geq 0}$, and

$$E_y L_{\tau_A}(z) = g_A(y,z) = \frac{R(y,A)+R(z,A)-R_A(y,z)}{2},$$

can establish via Markov's inequality a general estimate of the form:

$$\sup_{x\in F} P_x\left(\sup_{s\leq t} R(x,X_s)\geq \varepsilon\right) \leq \frac{32N(F,\varepsilon/4)}{\varepsilon}\left(\delta + \frac{t}{\inf_{x\in F}\mu(B_R(x,\delta))}\right),$$

where $N(F,\varepsilon)$ is the minimal size of an ε cover of F.

Metric measure convergence $\Rightarrow$ estimate holds uniformly in n $\Rightarrow$ tightness (application of Aldous' tightness criterion).

Similar estimate also gives non-explosion in locally compact case.

4. APPLICATIONS

TREES

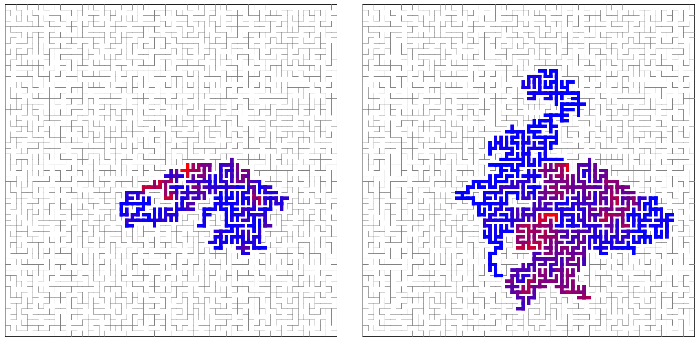
For any sequence of graph trees $(T_n)_{n\geq 1}$ such that

$$(V(T_n), a_n R_n, b_n \mu_n) \to (\mathcal{T}, R, \mu)\,,$$

it holds that

$$\left(a_n^{-1} X_{t a_n b_n}\right)_{t\geq 0} \to (X_t)_{t\geq 0}\,.$$

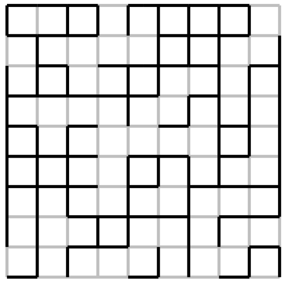
- Critical Galton-Watson trees with finite variance conditioned on size, $a_n = n^{1/2}$, $b_n = n$.
- Uniform spanning tree in two dimensions, $a_n = n^{5/4}$, $b_n = n^2$, e.g. after 5,000 and 50,000 steps (picture: Sunil Chhita).



- Many other interesting models...

CONJECTURE FOR CRITICAL PERCOLATION

Bond percolation on integer lattice $\mathbb{Z}^d$:



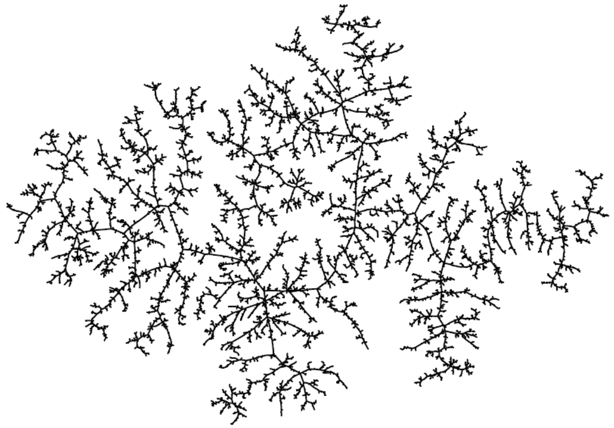
At criticality $p = p_c(d)$ in high dimensions, incipient infinite cluster (IIC) conjectured to have same scaling limit as Galton-Watson tree, e.g. [Hara/Slade 2000]. So, expect

$$\left(\mathrm{IIC}, n^{-2}R_{\mathrm{IIC}}, n^{-4}\mu_{\mathrm{IIC}}\right)$$

to converge, and thus obtain scaling limit for random walks. cf. recent work of [Ben Arous, Fribergh, Cabezas 2016] for branching random walk. NB. Diffusion scaling limit constructed in [C. 2009].

RANDOM WALK SCALING ON CRITICAL RANDOM GRAPH

Consider largest connected component $\mathcal{C}_1^n$ of $G(n, 1/n)$:



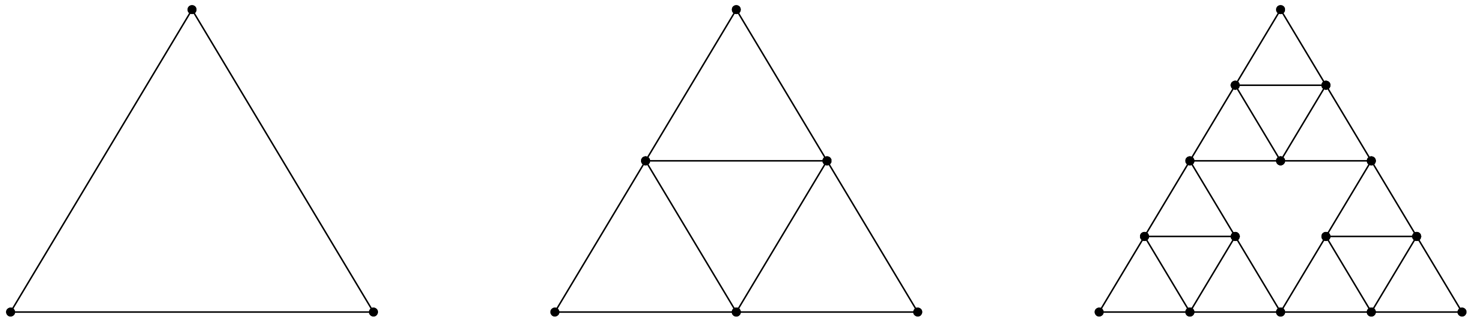
It holds that:

$$\left(\mathcal{C}_1^n, n^{-1/3}R_n, n^{-2/3}\mu_n\right) \to (F, R, \mu)\,,$$

cf. [Addario-Berry, Broutin, Goldschmidt 2012]. Hence, as in [C. 2012],

$$\left(n^{-1/3}X^n_{tn}\right)_{t\geq 0} \to (X_t)_{t\geq 0}\,.$$

HEAVY-TAILED RCM ON FRACTALS #1



Suppose that $P(c(x,y) \geq u) = u^{-\alpha}$ for $u \geq 1$ and some $\alpha \in (0,1)$. For gaskets, can then check that resistance homogenises [C., Hambly, Kumagai 2016]

$$\left(V_n, (3/5)^n R_n, 3^{-n}\mu_n\right) \to (F, R, \mu)\,,$$

where:
-(up to a deterministic constant) R is the standard resistance,
- μ is a Hausdorff measure on fractal.
Hence VSRW converges to Brownian motion (spatial scaling assumes graphs already embedded into limiting fractal):

$$(X^n_{t5^n})_{t\geq 0} \to (X_t)_{t\geq 0}\,.$$

HEAVY-TAILED RCM ON FRACTALS #2

It further holds that

$$\nu_n := 3^{-n/\alpha} \sum_{x \in V_n} c(x)\delta_x \to \nu = \sum_i v_i \delta_{x_i},$$

in distribution, where $\{(v_i, x_i)\}$ is a Poisson point process with intensity $cv^{-1-\alpha} dv \mu(dx)$. Hence CSRW (and discrete time random walk) converges:

$$\left(X^{n,\nu_n}_{t(5/3)^n 3^{n/\alpha}} \right)_{t \geq 0} \to (X^\nu_t)_{t \geq 0},$$

where the limiting process X^ν is the **Fontes-Isopi-Newman (FIN)** diffusion on the limiting fractal.

Similarly scaling result for heavy-tailed Bouchaud trap model.

HEAT KERNEL ESTIMATES FOR FIN DIFFUSION

If $\mu(B_d(x,r)) \asymp r^{d_f}$, $d(x,y) \asymp R(x,y)^{\beta}$, then heat kernel is of form:

$$\mathbf{E}\left(p_t^{\nu}(x,y)\right) \asymp c_1 t^{-d_s/2} \exp\left\{-c_2\left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right\},$$

where

$$d_w := \frac{d_f}{\alpha} + \frac{1}{\beta}, \qquad d_s := \frac{2d_f}{\alpha d_w}.$$

NB. Can only prove for $\alpha > \alpha_c$.

Almost-surely (for any $\alpha \in (0,1)$):
Log fluctuations above diagonal locally ($O(1)$ globally);
Loglog fluctuations below diagonal locally (log globally);
No fluctuations in off-diagonal decay term.